

Maximal Entanglement and Teleportation using Arthurs-Kelly Interaction for Qubits

Subhayan Sahu^{1,*} and S. M. Roy^{2,†}

¹Undergraduate Department, Indian Institute of Science, Bangalore

²HBCSE, Tata Institute of Fundamental Research, Mumbai

We study entanglement generation between a system qubit and three apparatus qubits using an exactly soluble Arthurs-Kelly type model. We demonstrate the possibility of generating an EPR-like maximally entangled system-apparatus state, in which the second qubit of the usual EPR state is replaced by a three qubit state. We design a very simple teleportation protocol to transfer the unknown state of the system onto one of the apparatus qubits which can then be teleported via a quantum channel.

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Introduction. The idea of Quantum entanglement was introduced by Schrödinger [1]. The truly non-classical features of entanglement drew wide attention after the work of Einstein, Podolsky and Rosen (EPR) [2] and John Bell [3]. In the last two decades, quantum entanglement has emerged as an important resource for Quantum Information processing tasks, such as Quantum Teleportation [4–7], Quantum Key Distribution [8], Quantum Computing [9] and Quantum Metrology [10]. Here we report a very simple method for teleportation of an unknown state of a system qubit using an interaction that maximally entangles the qubit with three apparatus qubits.

Entangling interactions have been used previously in quantum measurement theory. Von Neumann introduced the idea of tracking of a system observable by using an apparatus observable [11]: the system interacts with the apparatus for some time such that the apparatus observable has the same expectation value in the final state as the system observable in the initial state. The idea was extended by Arthurs and Kelly [12] to the joint tracking of two canonically conjugate observables by two commuting apparatus observables. The tracking cannot be noiseless and hence, only approximate joint measurement of non-commuting system observables is possible. This yields a joint measurement uncertainty relation [13, 14] for conjugate observables. It has also been shown that the Arthurs Kelly (AK) interaction can be utilised for remote quantum tomography of continuous variable systems [15]. Extensions of Arthurs-Kelly type measurements have been made to joint measurement of different components of spin observables [16–18].

Here we consider the Levine et al [16] AK-type measurement interaction between a system qubit and three apparatus qubits such that the three (mutually non-commuting) spin components of the system qubit are tracked by mutually commuting spin components of the

apparatus qubits. We (i) derive a joint measurement uncertainty relation, (ii) show that the interaction can give rise to maximal entanglement generation between an unknown system qubit and the apparatus, and (iii) utilise the maximal entanglement to devise a new protocol to teleport the unknown state of the system qubit.

Joint Measurement Uncertainty Relation in an Arthurs Kelly Type Measurement of Spin Components. In the model of Levine et al [16] the three non-commuting spin components of a spin half particle P are coupled with three meter qubits (A_1, A_2 and A_3) via a magnetic interaction,

$$H = K(\sigma_x^P \sigma_z^{A_1} + \sigma_y^P \sigma_z^{A_2} + \sigma_z^P \sigma_z^{A_3}) = K \sum_{i=1}^3 \sigma_i^P \sigma_z^{A_i} \quad (1)$$

where σ_i^Q is the i^{th} Pauli Operator for the particle Q. Neglecting other interactions during the short interaction time T , the unitary evolution of the four qubit initial state $|0\rangle$ to the final state $|\mathbf{T}\rangle$ is given by,

$$|\mathbf{T}\rangle = \hat{U}|0\rangle, \quad \hat{U} = \exp[-iHT]. \quad (2)$$

The unitary evolution operator can be simplified to give

$$\hat{U} = \cos \theta \mathbf{1} - \frac{i}{\sqrt{3}} \sin(\theta) \sum_{i=1}^3 \sigma_i^P \sigma_z^{A_i} \quad (3)$$

where $\mathbf{1}$ denotes the identity operator and $\theta = \sqrt{3}KT$. The time evolved meter operators after time T in the Heisenberg picture can be written as

$$\begin{aligned} \sigma_x^{A_i}(T) &= \hat{U}^\dagger \sigma_x^{A_i} \hat{U} = \cos^2 \theta \sigma_x^{A_i} - \sin(2\theta) \sigma_i^P \sigma_y^{A_i} / \sqrt{3} \\ &\quad + \frac{1}{3} \sin^2 \theta (\sigma_x^{A_i} + 2\sigma_y^{A_i} \sum_{j,k=1}^3 \epsilon_{ijk} \sigma_j^P \sigma_z^{A_k}), \end{aligned} \quad (4)$$

where ϵ_{ijk} is the totally antisymmetric symbol with $\epsilon_{123} = 1$, and $\sigma_1^P = \sigma_x^P, \sigma_2^P = \sigma_y^P, \sigma_3^P = \sigma_z^P$; this yields,

*Electronic address: subhayan@ug.iisc.in

†Electronic address: smroy@hbcse.tifr.res.in

for example,

$$\langle \mathbf{T} | \sigma_x^{A_1} | \mathbf{T} \rangle = \langle \mathbf{0} | \cos^2 \theta \sigma_x^{A_1} - \sin(2\theta) \sigma_1^P \sigma_y^{A_1} / \sqrt{3} + \frac{1}{3} \sin^2 \theta (\sigma_x^{A_1} + 2\sigma_y^{A_1} (\sigma_2^P \sigma_z^{A_3} - \sigma_3^P \sigma_z^{A_2})) | \mathbf{0} \rangle. \quad (5)$$

If we start with an initial state,

$$|\mathbf{0}\rangle = |\psi\rangle |+\rangle^{A_1} |+\rangle^{A_2} |+\rangle^{A_3} \equiv |\psi, +, +, +\rangle \quad (6)$$

where $|\psi\rangle$ is the unknown initial state of particle P , $|\pm\rangle$ are eigenstates of the Pauli Matrix σ_y with eigenvalues ± 1 ,

$$\sigma_y |\pm\rangle = \pm |\pm\rangle, \quad \sigma_z |+\rangle = |-\rangle, \quad (7)$$

we obtain,

$$\langle \mathbf{0} | \Sigma_i | \mathbf{0} \rangle = \langle \psi | \sigma_i^P | \psi \rangle, \quad (8)$$

where,

$$\Sigma_i \equiv -\frac{\sqrt{3}}{\sin(2\theta)} \sigma_x^{A_i}(T). \quad (9)$$

For the variances,

$$\begin{aligned} (\Delta \sigma_i^P)^2 &= \langle \psi | (\sigma_i^P)^2 | \psi \rangle - \langle \psi | \sigma_i^P | \psi \rangle^2 \\ (\Delta \Sigma_i)^2 &= \langle \mathbf{0} | \Sigma_i^2 | \mathbf{0} \rangle - \langle \mathbf{0} | \Sigma_i | \mathbf{0} \rangle^2 \end{aligned} \quad (10)$$

we have the uncertainty relations,

$$\begin{aligned} (\Delta \Sigma_i)^2 - (\Delta \sigma_i^P)^2 &= \frac{3}{\sin^2(2\theta)} - 1 \geq 2 \\ \sum_{i=1}^3 (\Delta \sigma_i^P)^2 &= 2, \\ \sum_{i=1}^3 (\Delta \Sigma_i)^2 &= \frac{9}{\sin^2(2\theta)} - 1 \geq 8. \end{aligned} \quad (11)$$

The tracking of σ_i^P by Σ_i is not noiseless; the minimum noise is achieved for $\theta = \pi/4$.

Entanglement generation. The time evolved state after time T is,

$$\begin{aligned} |\mathbf{T}\rangle &= \cos \theta |\psi, +, +, +\rangle - \frac{i \sin \theta}{\sqrt{3}} \left(\sigma_x^P |\psi, -, +, +\rangle \right. \\ &\quad \left. + \sigma_y^P |\psi, +, -, +\rangle + \sigma_z^P |\psi, +, +, -\rangle \right). \end{aligned} \quad (12)$$

If the system qubit is denoted by

$$|\psi\rangle = \begin{pmatrix} a \\ b \end{pmatrix} = a|0\rangle + b|1\rangle, \quad (13)$$

the above state can be expressed in the following form:

$$\begin{aligned} |\mathbf{T}\rangle &= |0\rangle \left(a \cos \theta |+++\rangle - i(\sin \theta / \sqrt{3}) \right. \\ &\quad \left. \times (b | -++\rangle - ib | +-+\rangle + a | ++-\rangle) \right) \\ &\quad + |1\rangle \left(b \cos \theta |+++\rangle - i(\sin \theta / \sqrt{3}) \right. \\ &\quad \left. \times (a | -++\rangle + ia | +-+\rangle - b | ++-\rangle) \right). \end{aligned} \quad (14)$$

Note that the apparatus states multiplying the system states $|0\rangle$ and $|1\rangle$ are mutually orthogonal if and only if $\cos^2 \theta = 1/4$; in that case the above state is expressed in the Schmidt biorthogonal form [19].

The final reduced density matrices for the system qubit P and the three-qubit apparatus $A = A_1, A_2, A_3$ are,

$$\rho^P = \text{Tr}_{\{A_1, A_2, A_3\}} |\mathbf{T}\rangle \langle \mathbf{T}|; \quad \rho^A = \text{Tr}_{(P)} |\mathbf{T}\rangle \langle \mathbf{T}|. \quad (15)$$

This yields,

$$\rho_{00}^P = \cos^2 \theta |a|^2 + \frac{1}{3} \sin^2 \theta (1 + |b|^2)$$

$$\rho_{01}^P = (\rho_{10}^P)^* = (\cos^2 \theta - \frac{1}{3} \sin^2 \theta) ab^*$$

$$\rho_{11}^P = \cos^2 \theta |b|^2 + \frac{1}{3} \sin^2 \theta (1 + |a|^2)$$

The system-apparatus entanglement E is given by the von Neumann entropy of the reduced density matrix of system P or equivalently of the apparatus A , [20]

$$\begin{aligned} E &= -\text{Tr} \rho^P \ln \rho^P = -\text{Tr} \rho^A \ln \rho^A \\ &= -\lambda \ln \lambda - (1 - \lambda) \ln (1 - \lambda), \end{aligned} \quad (16)$$

where λ and $1 - \lambda$ are eigenvalues of ρ^P which obey

$$\lambda(1 - \lambda) = \det \rho^P = \frac{2}{9} \sin^2 \theta (1 + 2 \cos^2 \theta).$$

The Entanglement is maximum when $\lambda = 1/2$, i.e. $\cos^2 \theta = 1/4$,

$$E \leq \ln 2; \quad E = \ln 2 \text{ for } \cos^2 \theta = 1/4. \quad (17)$$

For $\cos \theta = 1/2$, $\sin \theta = \pm \sqrt{3}/2$, the corresponding maximally entangled final states $|\mathbf{T}\pm\rangle$ assume the Schmidt biorthogonal forms,

$$\begin{aligned} |\mathbf{T}\pm\rangle &= \frac{|0\rangle}{2} \left(a |+++\rangle \mp i(b | -++\rangle \right. \\ &\quad \left. - ib | +-+\rangle + a | ++-\rangle) \right) \\ &\quad + \frac{|1\rangle}{2} \left(b |+++\rangle \mp i(a | -++\rangle \right. \\ &\quad \left. + ia | +-+\rangle - b | ++-\rangle) \right). \end{aligned} \quad (18)$$

These states, are analogous to the two qubit EPR states, with one of the qubits replaced by three qubits.

Since $\theta = \sqrt{3}KT$, it is seen that by varying the product of the strength and duration of interaction, such that $\cos \theta = 1/2$, $\sin \theta = \pm \sqrt{3}/2$, maximal entanglement between the system and the apparatus can be achieved.

Teleportation.

If we make measurements to project the states $|\mathbf{T}\pm\rangle$ on to the sub-spaces $|0\rangle |+\rangle^{A_1} |+\rangle^{A_3}$, and $|0\rangle |+\rangle^{A_1} |+\rangle^{A_3}$,

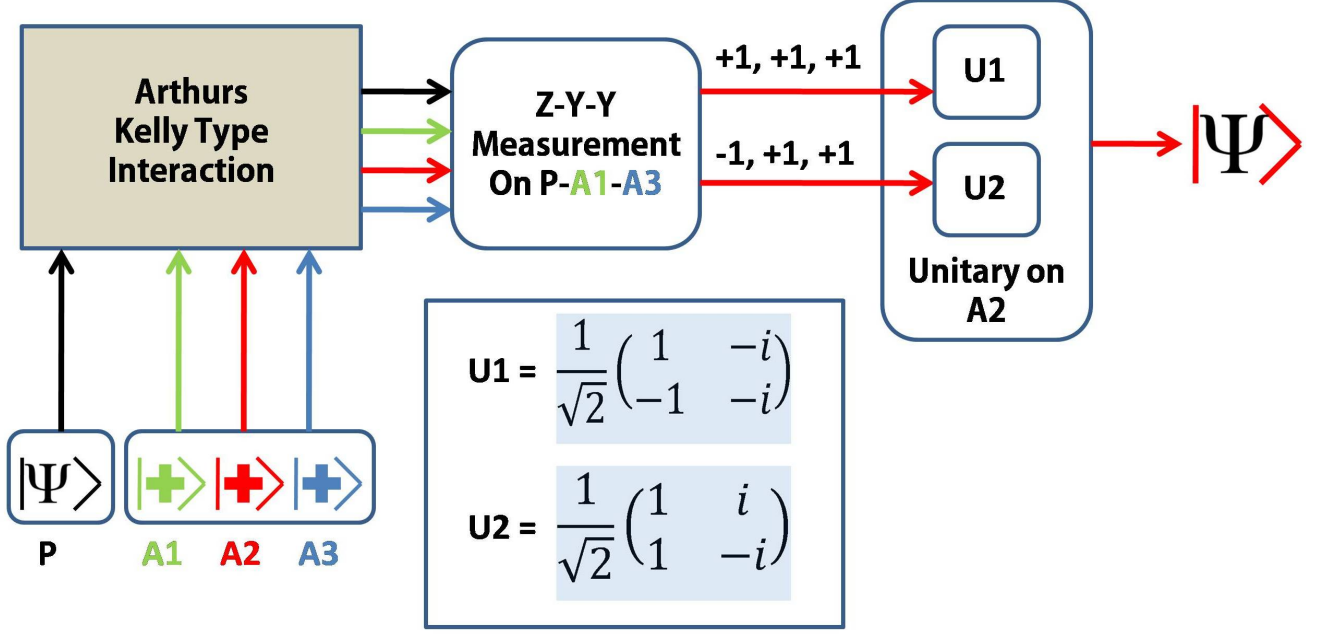


FIG. 1: A schematic diagram showing the new teleportation protocol. The qubit in black is the system qubit P in an unknown state, and the qubits in green, blue and red are suitably prepared apparatus qubits A_1, A_2 and A_3 respectively. After an Arthurs-Kelly type interaction, commuting spin components $\sigma_z^P = Z^P, \sigma_y^{A_1} = Y^{A_1}, \sigma_y^{A_3} = Y^{A_3}$ are measured. The qubit A_2 corresponding to the results $+, +, +$ and $-, +, +$ of the measurements go into separate quantum channels through which it can be teleported to a remote location. If the parameter θ determined by the interaction obeys $\cos\theta = 1/2, \sin\theta = \sqrt{3}/2$, the unknown state of P can be recovered by applying the unitary transformation U_1 on A_2 arriving through the first channel, or U_2 on A_2 arriving through the second channel. If $\cos\theta = 1/2, \sin\theta = -\sqrt{3}/2$, the required unitary transformations are $\sigma_z U_1$ and $\sigma_z U_2$ for the two channels (not shown in the figure). The unitary transformation can be applied either before or after sending the particle A_2 to the distant location.

we obtain respectively the following normalized states of the qubit A_2 ,

$$\begin{aligned} 2\langle 0 | \langle + |^{A_1} \langle + |^{A_3} | \mathbf{T} \pm \rangle &= a | + \rangle^{A_2} \mp b | - \rangle^{A_2} \\ &= -i U_1^\dagger (a | 0 \rangle \pm b | 1 \rangle)^{A_2} \\ 2\langle 1 | \langle + |^{A_1} \langle + |^{A_3} | \mathbf{T} \pm \rangle &= b | + \rangle^{A_2} \pm a | - \rangle^{A_2} \\ &= \pm i U_2^\dagger (a | 0 \rangle \pm b | 1 \rangle)^{A_2}, \end{aligned} \quad (19)$$

where U_1 and U_2 are the unitary transformations

$$U_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -i \\ -1 & -i \end{pmatrix}; U_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ 1 & -i \end{pmatrix} \quad (20)$$

We subject the state to measurements of $\sigma_z^P = Z^P, \sigma_y^{A_1} = Y^{A_1}, \sigma_y^{A_3} = Y^{A_3}$. A flowchart is provided in Fig1. The unmeasured qubit A_2 is sent into one quantum channel if the measurement of the other qubits gives the results $+, +, +$, and to a second quantum channel if the results are $-, +, +$, and discarded otherwise. Then A_2 is left in a superposition state given by Eqn. (19). Hence the apparatus qubit A_2 can be unitarily transformed to the unknown system state of P by applying the unitary transformations U_1 and U_2 for the two channels, if $\cos\theta = 1/2, \sin\theta = \sqrt{3}/2$; the required unitary transformations are respectively $\sigma_z U_1$ and $\sigma_z U_2$ if $\cos\theta = 1/2,$

$\sin\theta = -\sqrt{3}/2$. The qubit A_2 is then teleported to a distant location through a quantum channel. The unitary transformation required to retrieve the unknown initial state can be applied on qubit A_2 either before or after teleportation.

Entanglement swapping is also easy. Suppose that some other qubit Q was entangled with the system qubit P ; then after the full protocol has been performed, Q is entangled to qubit A_2 .

Comparison with usual teleportation protocols.

A conventional Quantum Teleportation scheme[4] has 4 main steps: (i) An EPR pair $E1, E2$ is shared by Alice and Bob at distant locations. (ii) The system particle P with unknown state is received by Alice and she makes a Bell-state measurement on the joint state of that particle and $E1$ and (iii) communicates the result via a classical channel to Bob; (iv) Bob then makes a unitary transformation on $E2$ depending on the classical information to replicate the unknown system state. In the alternative Teleportation scheme reported here, the steps of EPR-pair sharing, Bell projection and classical communication are not necessary; instead, the Arthurs-Kelly entangling interaction and single particle spin measurements are used to “transfer” an unitary transform of the unknown state to

an apparatus qubit. The unknown state can be recovered from the apparatus qubit by applying the inverse unitary transform, either before or after teleportation of that qubit. One advantage of the present scheme, apart from not needing EPR-sharing, is that single particle spin measurements are much easier than Bell state measurements. It will therefore be worthwhile to attempt experimental realisation of the Levine et al [16] Arthurs Kelly type interaction of four qubits which is the starting point of our protocol.

Conclusions. We have shown that the Levine et al

[16] Arthurs Kelly type interaction can generate maximal entanglement between a system qubit and three apparatus qubits. We utilise this to introduce a novel scheme of teleportation which has some advantages over the conventional methods. The practical realisation of the four qubit interaction is an interesting challenge.

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